

CRITERIA FOR STABILITY IN CIRCUITS CONTAINING NON-LINEAR RESISTANCE

BY

LEO VICTOR SKINNER

B.S., University of Missouri, 1930

M.S., University of California, 1935

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
ENGINEERING IN THE GRADUATE COLLEGE
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Many practical problems arise involving non-linear resistance which cannot be solved by linear circuit theory. Examples are certain circuits containing electron tubes, arcs, or gaseous discharges. Analogous problems arise in connection with mechanical vibrations.

Non-linear circuits of one degree of freedom have been rather thoroughly studied by van der Pol and others. An excellent treatment of this subject is given in a treatise on "Non-Linear Mechanics" by Minorsky. Very little has been published on non-linear circuits of two or more degrees of freedom. In this paper circuits of two and three degrees of freedom containing a non-linear resistance for which the voltage current characteristic is representable by a fourth degree poly-

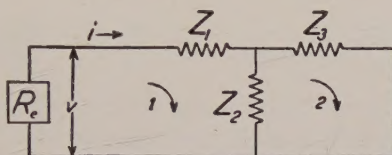


Fig. 1 — General quasi-linear circuit of two degrees of freedom
in terms of mesh currents.

nomial will be treated. Conditions will be established under which self-excitation may occur at each of the resonant frequencies. It will be shown that for the fourth degree characteristic assumed that only one of the resonant frequencies may be present at a time.

Figure 1 represents the most generally high oscillatory circuit of two degrees of freedom, in terms of the mesh currents i_1 and i_2 , containing a non-linear resistance R_e in the first mesh. If Kirchoff's second law is applied to the circuit of Fig. 1, a system of two second order non-linear differential equations result in which the instantaneous mesh currents i_1 and i_2 are the dependent variables and t is the independent variable. This system of equations may be solved by applying a combination of the method of variation of parameters and the perturbation method. In this manner the two second order equations are reduced to four first order equations by successive approximations. Since the circuit is assumed to be highly oscillatory, i.e., the resistances are much smaller than the reactances, a first order approximation will usually give sufficiently accurate results.

The voltage-current characteristic of R_e is assumed to be of the form

$$v = \alpha i_1 + \beta i_1^2 + \gamma i_1^3 + \epsilon i_1^4. \quad (1)$$

The current in the first mesh is shown to be of the form

$$i_1 = a \cos (\Omega_1 t + \phi_1) + b \cos (\Omega_2 t + \phi_2). \quad (2)$$

If i_1 from (2) is substituted into (1), there results

$$v = R_{e1} a \cos (\Omega_1 t + \phi_1) + R_{e2} b \cos (\Omega_2 t + \phi_2) + [\text{combination frequency terms}], \quad (3)$$

where

$$R_{e1} = \alpha + \frac{3\gamma}{4} (a^2 + 2b^2) \quad R_{e2} = \alpha + \frac{3\gamma}{4} (b^2 + 2a^2). \quad (4)$$

It is assumed that none of the combination frequencies are at or near Ω_1 or Ω_2 . On this basis the response of the circuit to terms involving combination frequencies will be small and they may be neglected.

To the first order approximation the resonant frequencies Ω_1 and Ω_2 are the same as for the case where all resistances are neglected. The system of equations for the first order determination of the amplitudes a and b are

$$\frac{da^2}{dt} = -\frac{2\delta_{o1}a^2}{R_{o1}} [R_{o1} + R_{e1}] \quad \frac{db^2}{dt} = -\frac{2\delta_{o2}b^2}{R_{o2}} [R_{o2} + R_{e2}] \quad (5)$$

where δ_{o1} and δ_{o2} are the logarithmic decrements of the linear circuit at Ω_1 and Ω_2 respectively and R_{o1} and R_{o2} are the input resistances to the linear circuit at Ω_1 and Ω_2 respectively. In view of (4) equations (5) may be written

$$\frac{da^2}{dt} = -\frac{3\gamma\delta_{o1}a^2}{2R_{o1}} \left[\frac{4}{3\gamma} (R_{o1} + \alpha) + a^2 + 2b^2 \right] \quad (6)$$

$$\frac{db^2}{dt} = -\frac{3\gamma\delta_{o2}b^2}{2R_{o2}} \left[\frac{4}{3\gamma} (R_{o2} + \alpha) + b^2 + 2a^2 \right].$$

The variables cannot be separated in equations (6) to give a transient solution for the amplitudes. However, it may be noted that when

$\frac{da^2}{dt} = \frac{db^2}{dt} = 0$, a^2 and b^2 take on equilibrium values which will be denoted by a_s^2 and b_s^2 respectively. Equations (6) reduce to

$$a_s^2 \left[a_s^2 + 2b_s^2 + \frac{4}{3\gamma} (R_{o1} + \alpha) \right] = 0 \quad (7)$$

$$b_s^2 \left[b_s^2 + 2a_s^2 + \frac{4}{3\gamma} (R_{o2} + \alpha) \right] = 0.$$

The four solutions of (7) are $a_s^2 = 0$, $b_s^2 = 0$; $a_s^2 = \frac{1}{3} (2b_m^2 - a_m^2)$, $b_s^2 = \frac{1}{3} (2a_m^2 - b_m^2)$; $a_s^2 = a_m^2$, $b_s^2 = 0$; and $a_s^2 = 0$, $b_s^2 = b_m^2$,

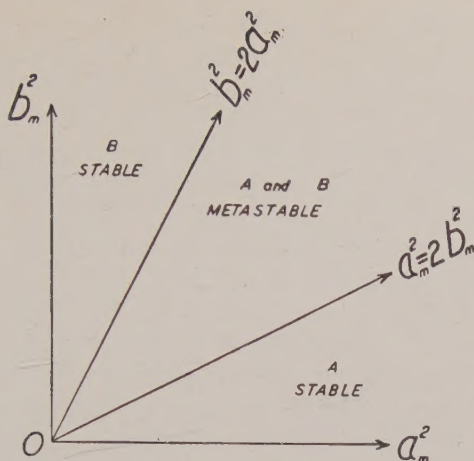


Fig. 2—Stability diagram for general quasi-linear circuit of two degrees of freedom in terms of mesh currents.

where

$$a_m^2 = -\frac{4}{3\gamma} (R_{o1} + \alpha) \quad b_m^2 = -\frac{4}{3\gamma} (R_{o2} + \alpha). \quad (8)$$

A second perturbation procedure is employed to determine the conditions under which each solution may be stable. This procedure consists in displacing a^2 and b^2 by small amounts from their equilibrium values a_s^2 and b_s^2 respectively and determining the conditions under which the circuit will return to the same equilibrium values. By applying this method it is found that the solution $a_s^2 = 0$, $b_s^2 = 0$ will be stable if $R_{o1} + \alpha > 0$ and $R_{o2} + \alpha > 0$. This solution corresponds to no oscillations at either frequency and is of little interest.

The second solution corresponds to oscillations at both frequencies at the same time. The condition for stability is $\frac{\gamma^2 \delta_{o1} \delta_{o2}}{R_{o1} R_{o2}} a_s^2 b_s^2 < 0$. Since all quantities on the left hand side of the inequality are positive this solution can never be stable.

The last two solutions correspond to oscillations at one of the two resonant frequencies. The results for these cases are summarized by the a_m^2 , b_m^2 diagram of Fig. 2. Since a_m^2 and b_m^2 correspond to squares of amplitudes it is necessary to consider only the first quadrant of the diagram. The quadrant is divided into regions by the lines $a_m^2 = 2b_m^2$ and $b_m^2 = 2a_m^2$. The modes of oscillation that may be stable in each region are indicated. In the center region either mode A or mode B may be present but not both. To use the diagram a_m^2 and b_m^2 are

calculated from relations (8) for a specific set of circuit parameters, and the corresponding point is plotted on the diagram. If the point falls in the region marked *A* Stable, then only the first resonant frequency may occur. If the point falls in the region marked *B* Stable, then only the second resonant frequency may occur. If the point falls in the region marked *A* and *B* metastable, then either frequency may occur but not both.

The point (a_m^2 , b_m^2) may be made to move on the diagram by adjusting the circuit parameters. Let it be assumed that the point is in the *A* Stable region. If it moves over into the center region the *A*-mode will continue to exist. However, if the point moves over into the *B* Stable region, the amplitude and frequency will abruptly change to that of the *B*-mode. Now if the point is made to retrace its path, the *B*-mode will continue until the point crosses the line into the *A* Stable region where the amplitude and frequency will abruptly change to that of the *A*-mode. There is thus a hysteresis effect associated with the transition from one frequency to the other.

A highly oscillatory circuit of three degrees of freedom containing a non-linear resistance in one of the meshes may be treated in the same manner as the circuit just considered. Only the results will be presented here. The voltage-current characteristic is assumed to be the polynomial (1). The conditions for no oscillations in the circuit are $R_{o1} + \alpha > 0$, $R_{o2} + \alpha > 0$, and $R_{o3} + \alpha > 0$, where R_{o3} is the input resistance to the linear circuit at the third resonant frequency Ω_3 . The presence of two or all three of the resonant frequencies at a time is shown to be unstable. The conditions under which each of the resonant frequencies may exist alone are illustrated on the a_m^2 , b_m^2 , c_m^2 diagram of Fig. 3. In the figure a cube of arbitrary size is constructed on the positive a_m^2 , b_m^2 , c_m^2 axes. The planes $a_m^2 = 2b_m^2$, $b_m^2 = 2a_m^2$, $b_m^2 = 2c_m^2$, $c_m^2 = 2b_m^2$, $c_m^2 = 2a_m^2$, and $a_m^2 = 2c_m^2$ divide each of the three faces of the cube opposite from the origin into four equal squares. The line determined by the origin and the point

(a_m^2 , b_m^2 , c_m^2), calculated from (8) and the relation $c_m^2 = -\frac{4}{3\gamma} (R_{o3} + \alpha)$,

will in general pass through one of those squares. If the line passes through a square where only one mode is indicated, then only that mode of oscillation may exist. If the line passes through a square where more than one mode is indicated, then any one but only one of the frequencies may exist.

A three dimensional hysteresis effect analogous to that for the previous circuit is present here. However, one situation arises here which could not occur for the two dimensional case. If the circuit is operating at one frequency under conditions such that any one of the

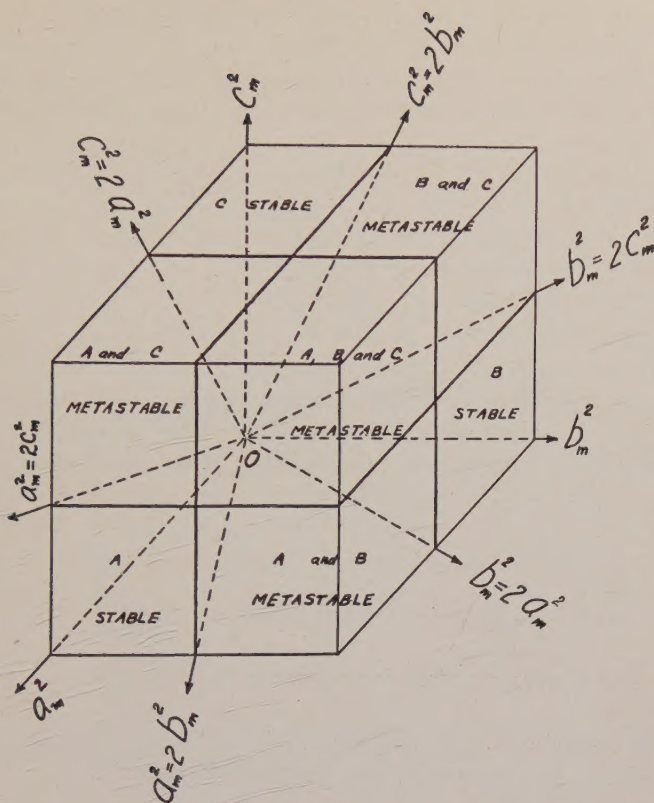


Fig. 3—Stability diagram for general quasi-linear circuit of three degrees of freedom in terms of mesh currents.

three frequencies are possible and the operating point is moved to a region where only the two frequencies not originally present are possible, the frequency of oscillation will change to that of the nearest stable region on the diagram.

A TWO-FREQUENCY OSCILLATOR

Self-excitation at two independent frequencies in a circuit of two or more degrees of freedom containing a non-linear element has been observed by several investigators. However, it appears that no satisfactory explanation of this phenomena has been given to date. For a highly oscillatory circuit of two degrees of freedom connected to a non-linear conductance G_e , as shown in Fig. 1, definite conditions are established in this paper for self-excitation to occur at one, the other, or both of the two resonant frequencies.

If Kirchoff's first law is applied to the circuit of Fig. 1, a system of two second order non-linear differential equations result, in which the instantaneous branch point voltages v_1 and v_2 are the dependent variables and t is the independent variable. This system of equations may be solved by applying a combination of the method of variation

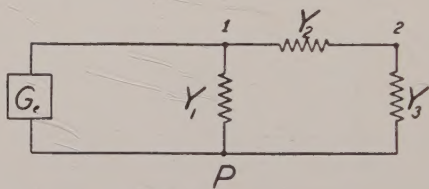


Fig. 1 — General quasi-linear circuit of two degrees of freedom in terms of branch-point voltages.

of parameters and the perturbation method. In this manner the two second order equations are reduced to four first order equations by successive approximations. Since the circuit is assumed to be highly oscillatory, i.e., the conductances are much smaller than the susceptances, a first order approximation will usually give sufficiently accurate results.

The current-voltage characteristic of G_e is assumed to be of the form

$$i = \alpha v_1 + \beta v_1^2 + \gamma v_1^3 + \epsilon v_1^4 + \rho v_1^5 + \sigma v_1^6. \quad (1)$$

The voltage at branch point 1 is shown to be of the form

$$v_1 = a \cos (\Omega_1 t + \phi_1) + b \cos (\Omega_2 t + \phi_2). \quad (2)$$

If v_1 from (2) is substituted into (1), there results

$$i = G_{e1} a \cos (\Omega_1 t + \phi_1) + G_{e2} b \cos (\Omega_2 t + \phi_2) + [\text{combination frequency terms}], \quad (3)$$

where

$$\begin{aligned} G_{e1} &= \alpha + \frac{3\gamma}{4}(a^2 + 2b^2) + \frac{5\rho}{8}(a^4 + 6a^2b^2 + 3b^4) \\ G_{e2} &= \alpha + \frac{3\gamma}{4}(b^2 + 2a^2) + \frac{5\rho}{8}(b^4 + 6b^2a^2 + 3a^4). \end{aligned} \quad (4)$$

It is assumed that none of the combination frequencies are at or near Ω_1 , or Ω_2 . On this basis the response of the circuit to terms involving combination frequencies will be small and they may be neglected.

To the first order approximation the resonant frequencies Ω_1 , and Ω_2 are the same as for the case where all conductances are neglected. The system of equations for the first order determination of the amplitudes a and b are

$$\frac{da^2}{dt} = -\frac{2\delta_{o1}a^2}{G_{o1}}[G_{o1} + G_{e1}] \quad \frac{db^2}{dt} = -\frac{2\delta_{o2}b^2}{G_{o2}}[G_{o2} + G_{e2}] \quad (5)$$

where δ_{o1} and δ_{o2} are the logarithmic decrements of the linear circuit at Ω_1 and Ω_2 respectively and G_{o1} and G_{o2} are the input conductances to the linear circuit at Ω_1 and Ω_2 respectively. In view of (4) equations (5) may be written

$$\begin{aligned} \frac{da^2}{dt} &= -\frac{\delta_{o1}a^2}{4G_{o1}}[8(G_{o1} + \alpha) + 6\gamma(a^2 + 2b^2) + 5\rho(a^4 + 6a^2b^2 + 3b^4)] \\ \frac{db^2}{dt} &= -\frac{\delta_{o2}b^2}{4G_{o2}}[8(G_{o2} + \alpha) + 6\gamma(b^2 + 2a^2) + 5\rho(b^4 + 6b^2a^2 + 3a^4)]. \end{aligned} \quad (6)$$

The variables cannot be separated in equations (6) to give a transient solution for the amplitudes. However, it may be noted that when

$\frac{da^2}{dt} = \frac{db^2}{dt} = 0$, a^2 and b^2 take on equilibrium values which will be denoted by x and y respectively. Equations (6) reduce to

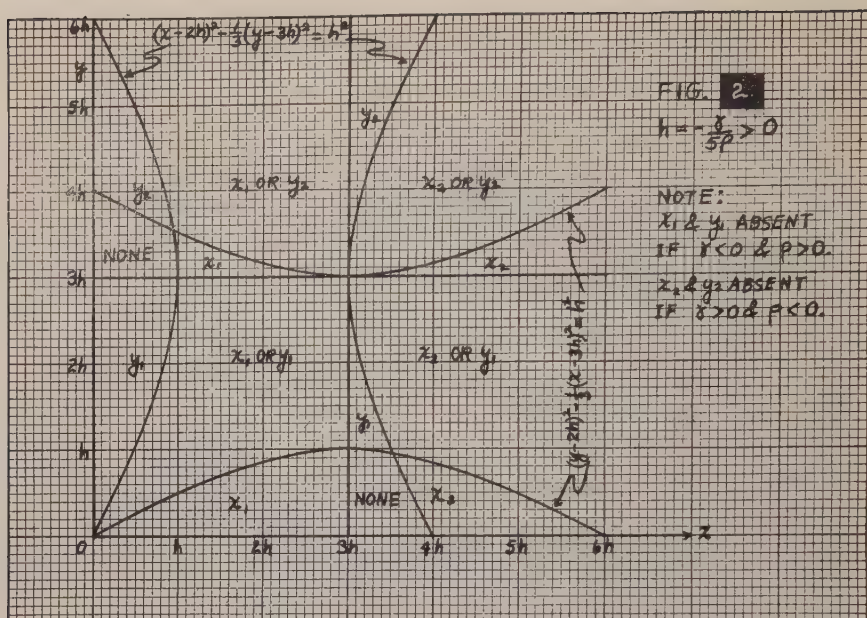
$$\begin{aligned} [5\rho(x^2 + 6xy + 3y^2) + 6\gamma(x + 2y) + 8(G_{o1} + \alpha)]x &= 0 \\ [5\rho(y^2 + 6yx + 3x^2) + 6\gamma(y + 2x) + 8(G_{o2} + \alpha)]y &= 0. \end{aligned} \quad (7)$$

The nine solutions of (7) are $x=0, y=0$; $x=x_1, y=0$; $x=x_2, y=0$; $x=0, y=y_1$; and $x=0, y=y_2$; where

$$\begin{aligned} \frac{x_1}{x_2} &= \frac{3\gamma}{5\rho} \left[-1 \pm \sqrt{1 - \frac{40\rho}{9\gamma^2}(G_{o1} + \alpha)} \right] \\ \frac{y_1}{y_2} &= \frac{3\gamma}{5\rho} \left[-1 \pm \sqrt{1 - \frac{40\rho}{9\gamma^2}(G_{o2} + \alpha)} \right] \end{aligned} \quad (8)$$

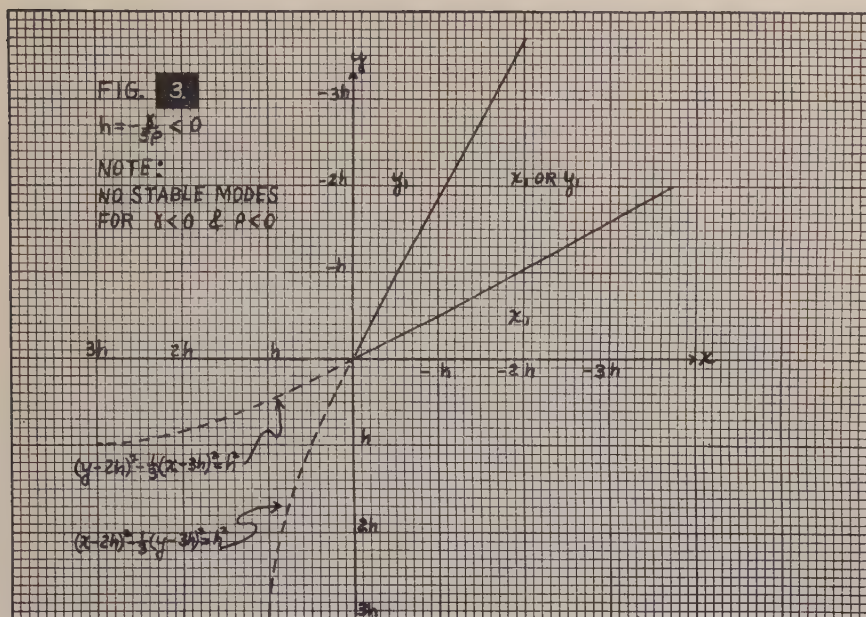
plus the four solutions of

$$\begin{aligned} 5\rho(x^2 + 6xy + 3y^2) + 6\gamma(x + 2y) + 8(G_{o1} + \alpha) &= 0 \\ 5\rho(y^2 + 6yx + 3x^2) + 6\gamma(y + 2x) + 8(G_{o2} + \alpha) &= 0. \end{aligned} \quad (9)$$



A second perturbation procedure is employed to determine the conditions under which each solution may be stable. This procedure consists in displacing a^2 and b^2 by small amounts from their equilibrium values x and y respectively and determining the conditions under which the circuit will return to the same equilibrium values. By applying this method it is found that the solution $x = 0$, $y = 0$ will be stable if $G_{o1} + \alpha > 0$ and $G_{o2} + \alpha > 0$. This solution corresponds to no oscillations at either frequency and is of little interest.

The next four solutions correspond to oscillations at one of the two resonant frequencies. The results for these cases are summarized by the x , y diagrams of Figs. 2 and 3. Since x and y correspond to squares of amplitudes it is necessary to consider only the first quadrant of the diagrams. Figure 2 is drawn for the condition that γ and ρ have opposite signs. The figure is divided into regions by two hyperbolas and two straight lines. The modes of oscillation that may be stable in each region are indicated. Where more than one mode is indicated only one may be present at a time. To use the diagram x and y are calculated from the relations (8) for a specific set of circuit parameters. Only positive values of x and y are considered. The point (x, y) is plotted on the diagram. If the point falls in a region marked " x ," then only the first resonant frequency may occur. If the point falls in a region

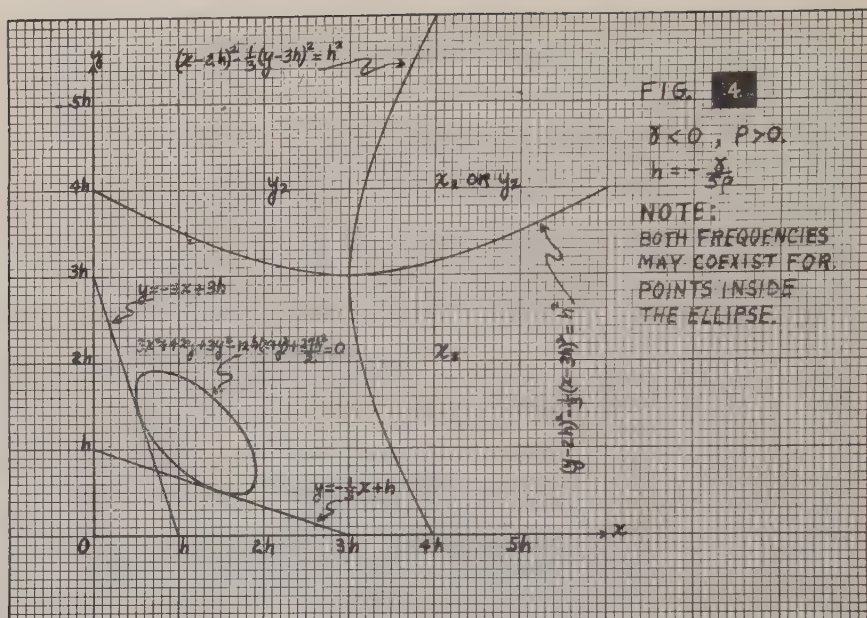


marked “y,” then only the second resonant frequency may occur. If the point falls in a region marked “x or y,” then either resonant frequency may occur but not both. It is to be noted that the x_1 and y_1 modes are absent if $\gamma < 0$ and $\rho < 0$, while the x_2 and y_2 -modes are absent if $\gamma > 0$ and $\rho < 0$.

The point (x, y) may be made to move on the diagram by adjusting the linear circuit parameters. Let it be assumed that the point is in the x_1 -region nearest the origin. If it moves across the hyperbola into the x_1 or y_1 -region, then the x_1 -mode will continue to exist. However, if the point continues across the other hyperbola to the y_1 -region, the amplitude and frequency will abruptly change to that for the y_1 -mode. Now if the point is made to retrace its path, the y_1 -mode will continue until the point moves across the hyperbola into the x_1 -region where the amplitude and frequency will abruptly change to that of the x_1 -mode. There is thus a hysteresis effect associated with the transition from one frequency to the other. A similar hysteresis effect is present for transitions between the x_2 and y_2 -modes.

Figure 3 is the x, y diagram for the conditions γ and ρ have the same signs. It is to be noted that the x_2 and y_2 -modes are absent for these conditions. The hysteresis effect is also present for this figure.

Figure 4 is an x, y diagram for $\gamma < 0$ and $\rho > 0$. Any solution (x, y)



of equations (9) which falls inside the ellipse of the figure will correspond to stable oscillations at both frequencies at the same time. The regions for which the x_2 and y_2 -modes may exist are the same here as shown on Fig. 2.

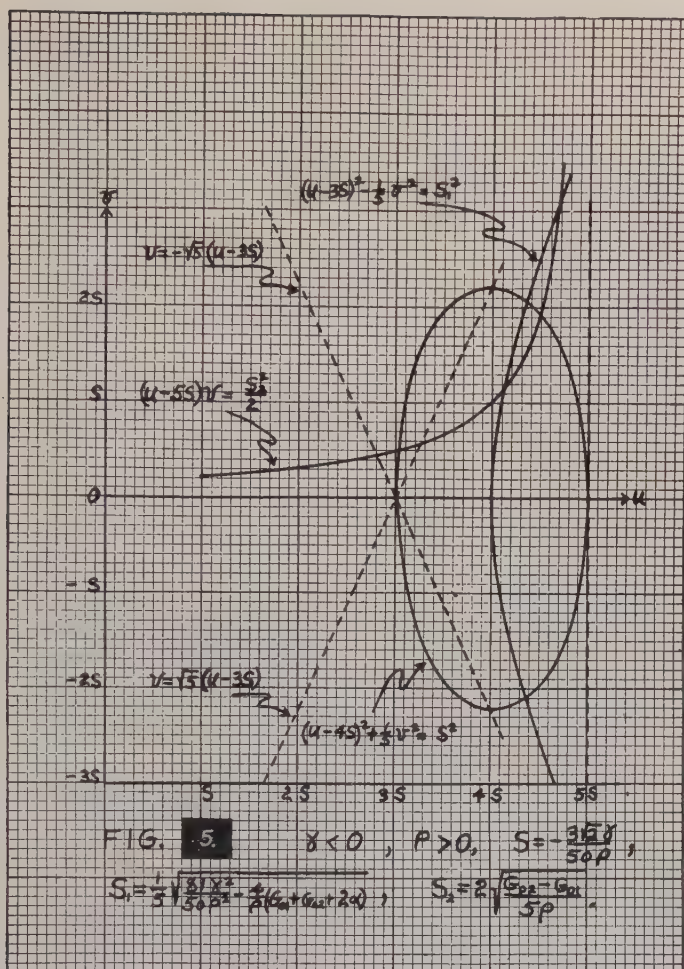
To establish the conditions under which a solution of (9) will correspond to a point inside the ellipse of Fig. 4 the u, v diagram of Fig. 5 is employed. Here the x, y axes of Fig. 4 have been rotated 45° in the positive direction to give the u, v axes. The intersection of the two hyperbolas shown on the figure gives a point (u, v) which corresponds to a solution of the equations (9). If the point falls inside the ellipse, then the solution will correspond to stable oscillations of the circuit at both frequencies. The locus of all points where the two hyperbolas are tangent is the ellipse. Hence there can be at most one intersection inside the ellipse. The conditions that one intersection occur are that

$$(S - \sqrt{S^2 + 2S_1^2}) (3S + \sqrt{S^2 + 2S_1^2})^3 < \frac{4}{5} S_2^4$$

$$< (S + \sqrt{S^2 + 2S_1^2}) (3S - \sqrt{S^2 + 2S_1^2})^3$$

where S, S_1 , and S_2 have the values indicated on Fig. 5.

The foregoing results have been verified at least qualitatively in the laboratory. The hysteresis effect on the transition from one resonant



frequency to the other has been definitely established. The coexistence of the two resonant frequencies has been observed many times. A more detailed experimental verification is now being made.

VITA

Leo Victor Skinner was born at Centralia, Missouri, on April 26, 1908. He received his elementary schooling in the public schools of that town. He received the Bachelor of Science degree from the University of Missouri in 1930 and the Master of Science degree from the University of California in 1935. An additional year of graduate work was completed at the University of California in 1937. Since June, 1946, he has been a graduate student at the University of Illinois.

His professional experience as an Electrical Engineer includes one and one-half years at Western Electric Company, Chicago, Ill.; two years at Photo Electric Research Corporation, New York, N. Y.; two years at Electro-Photo Engraver Company, Bloomfield, N. J.; and one and one-half years with the U.S. Navy at Camden, N. J. He has served as an officer with the U.S. Army and U.S. Air Force for eight years. At present, he is a Captain in the U.S. Air Force.

He is a member of Sigma Xi, Phi Kappa Phi, Tau Beta Pi, Eta Kappa Nu, Pi Mu Epsilon, associate member of American Institute of Electrical Engineers, and student member of the Institute of Radio Engineers.

